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L^p Contractive Projections and the Heat Semigroup for Differential Forms*

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A projection operator that is contractive on L^p for two distinct values of p is shown to be contractive for all values of p, and the range must be of a special form. This result is used to show that the heat semigroup for k-forms on many manifolds with nontrivial cohomology in dimension k cannot be contractive on any L^p for $p \neq 2$. © 1986 Academic Press, Inc.

1. INTRODUCTION

The main theme of this paper is that it is difficult for a projection operator to be contractive on L^p for more than one value of p. We will show in Section 2 that essentially all projections that are contractive on L^p for two distinct values of p must be contractive for all p, and the range space of the projection must essentially be closed under the operation $f(x) \to f(x)/|f(x)|$. This result holds not just for L^p spaces of functions, but also L^p sections of vector bundles with an inner product on the fibres.

The main motivation for studying these projections is that they yield immediate information about the heat semigroup for differential forms. One of the basic properties of the heat semigroup for functions is that it is contractive on L^p for all p, $1 \leqslant p \leqslant \infty$. In contrast, we will see in Section 3 that the heat semigroup for differential forms is often contractive only for $p = 2$. The idea is that if $e^{t\Delta}$ were contractive, then the Kodaira projection $\lim_{t \to \infty} e^{t\Delta}$ onto the harmonic k-forms would be a contractive projection, and the results of Section 2 apply.

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2. L^p Contractive Projections

Let X be a measure space with measure $d\mu$. We will consider the L^p spaces of complex- (or real-) valued functions on X, and more generally the L^p sections of a vector bundle over X equipped with an inner product on each fibre. Since we are not interested in the topology of the bundle we may assume it is globally trivialized, $X \times W$, where W is a finite-dimensional vector space and $\langle\ ,\ \rangle_x$ is an inner product on W depending measurably on x. By L^p we mean the space of (equivalence classes) of measurable functions from X to W such that

$$\|f\|_p^p = \int \langle f(x), f(x)\rangle_x^{p/2}\, d\mu < \infty.$$

With a little more effort we could allow W to be an infinite-dimensional Hilbert space, but we leave this generalization to the interested reader.

A bounded linear operator $T: L^p \to L^p$ is called a *projection* if $T^2 = T$, and it is called an L^p contraction if $\|Tf\|_p \leqslant \|f\|_p$. We will show that essentially any projection which is an L^p contraction for two different values of p must be of a very special form; in particular it will be an L^p contraction for all values of p, $1 \leqslant p \leqslant \infty$. We can think of this as a kind of "extrapolation" theorem, since it gets us outside the range of p's given by the familiar L^p interpolation theorems.

The key idea of the proof is that for a projection operator T, the contractive estimate is essentially equivalent to the "infinitesimal" version

$$\operatorname{Re} \int \langle Tf(x), f(x)\rangle_x\, |Tf(x)|_x^{p-2}\, d\mu(x) = \|Tf\|_p^p \tag{2.1}$$

(here $|Tf(x)|_x = \langle Tf(x), Tf(x)\rangle_x^{1/2}$). Note that for $p = 2$ this condition is equivalent to T being an orthogonal projection ($T = T^*$), and it is an easy exercise in plane geometry to show that a projection on L^2 is contractive if and only if it is orthogonal.

Lemma 2.1. *Let $T: L^p \to L^p$ be a projection. Then T is L^p contractive, for fixed p, $1 < p < \infty$, if and only if* (2.1) *holds for all $f \in L^p$.*

Proof. This is Proposition 6.2 of [2], since the norm is differentiable for $1 < p < \infty$.

Remark. If $p = 1$ there are simple examples of contractive projections not satisfying (2.1) (e.g., $T(x, y) = (x + ay, 0)$ for $|a| \leqslant 1$).

Theorem 2.2. *Suppose there exists a domain D which is dense in L^p for* *every p in $1 < p < \infty$, and let $T\colon D \to D$ be a linear operator which is a projection ($T^2 = T$) and which is an L^p contraction for two distinct values of p, say p_0 and p_1 with $p_0 < p_1$. Then T is an L^p contraction for all p, $1 \leqslant p \leqslant \infty$.*

Proof. By the M. Riesz interpolation theorem, T is an L^p contraction for all p satisfying $p_0 \leqslant p \leqslant p_1$. Fix $f \in D$ and consider the function

$$h(z) = \int |Tf(x)|_x^{z-2} \operatorname{Re}\langle Tf(x), f(x)\rangle_x \, d\mu$$

$$- \int |Tf(x)|_x^z \, d\mu.$$

This is an analytic function in z for $1 < \operatorname{Re} z < \infty$ because $f \in D$, and $h(z) = 0$ if z is real and $p_0 \leqslant z \leqslant p_1$ by Lemma 2.1. Thus $h(z) \equiv 0$, so T extends to a contraction on L^p for all p, $1 < p < \infty$. By passing to the limit we get the L^1 contractive estimate, and by duality, we get the L^∞ contractive estimate (here we are extending T to L^∞ by taking the adjoint of $T^*\colon L^1 \to L^1$; since D is not assumed dense in L^∞ this is a stronger result than $\|Tf\|_\infty \leqslant \|f\|_\infty$ for $f \in D$). Q.E.D.

Remark 1. The proof requires that we have the M. Riesz interpolation theorem with the optimal constant holding for L^p sections of a vector bundle. The usual proof due to Thorin goes through with trivial modifications (note that when the proof calls for taking simple functions $f(x)$ it really only requires that $|f(x)|_x$ be simple).

Remark 2. The hypothesis that T preserve a domain D dense in all L^p seems unlikely to be essential, and we conjecture that the theorem is true without it. On the other hand it is not difficult to verify for any reasonable operator.

COROLLARY 2.3. *Suppose the total measure is finite. If the operator T satisfies the hypotheses of the theorem, then on L^2 it is the orthogonal projection onto a closed subspace V with the property*

$$\text{if} \quad f \in V \qquad \text{then} \quad \frac{f(x)}{|f(x)|_x} \in V. \tag{2.2}$$

Proof. As already mentioned, the L^2 contractivity of a projection is equivalent to T being an orthogonal projection onto a closed subspace V.

Suppose T satisfies the hypotheses of the theorem. If $f \in V$ and $g \in V^\perp$ then

$$\int |f(x)|_x^{p-2} \langle f(x), g(x)\rangle_x \, d\mu = 0$$

for $1 < p \leqslant 2$ by (2.1), and so by letting $p \to 1$ we obtain

$$\int |f(x)|_x^{-1} \langle f(x), g(x) \rangle_x \, d\mu = 0. \tag{2.3}$$

This shows (2.2). Q.E.D.

Remark. There is also a converse result. If V is a closed subspace of L^2 satisfying (2.2), and if we also assume the total measure is finite, then the orthogonal projection T onto V is L^p contractive for all p, $1 \leqslant p \leqslant \infty$. To see this it suffices to prove $\|f+g\|_1 \geqslant \|f\|_1$ for all $f \in V$ and $g \in V^\perp$. But from (2.2) we obtain immediately (2.3) for $f \in V$ and $g \in V^\perp$, and the proof is completed by integrating the pointwise inequality

$$|f(x)+g(x)|_x \geqslant |f(x)|_x + \operatorname{Re} |f(x)|_x^{-1} \langle f(x), g(x) \rangle_x$$

(this pointwise inequality is established by expanding $|f(x)+g(x)|_x^2$ and using the inequality

$$|g(x)|_x \geqslant \operatorname{Re} |f(x)|_x^{-1} \langle f(x), g(x) \rangle_x).$$

THEOREM 2.4. *Suppose T and $I-T$ both satisfy the hypotheses of Theorem* 2.2. *Then*

$$Tf = \tfrac{1}{2}(f + Jf),$$

where J is an involution ($J^2 = I$) which is an isometry on all L^p. Furthermore J has the form

$$Jf(x) = h(x)\, f \circ \psi(x),$$

where h is a measurable function from X to the unitary operators on the fibre W, $\langle h(x)\,u, h(x)\,v \rangle_x = \langle u, v \rangle_x$ for all $u, v \in W$ and $h \circ \psi(x)\, h(x) = I$, and ψ is a measure preserving involution of the σ-field $\mathscr{F}$ of σ-finite measurable subsets of X modulo sets of measure zero, and $f \circ \psi$ is the induced map determined by $\chi_A \circ \psi = \chi_{\psi^{-1}(A)}$ for every measurable A with finite measure. If the measure space is assumed sufficiently regular (say isomorphic to an interval of $\mathbb{R}$) then ψ can be taken to be a measure preserving isomorphism $\psi: X \to X$ and $f \circ \psi$ the usual composition.

Proof. Set $J = 2T - I = T - (I - T)$. Since T is a projection, J is an involution, and

$$\|Jf\|_2^2 = \|Tf\|_2^2 + \|(I-T)\,f\|_2^2 = \|f\|_2^2$$

since T is orthogonal on L^2, so J is an isometry on L^2. We will show that J is also an isometry on L^4.

We write $f=f_1+f_2$ with $f_1=Tf$ and $f_2=(I-T)f$ so $Jf=f_1-f_2$. Now

$$\|Jf\|_4^4=\int (|f_1(x)|_x^2-2\operatorname{Re}\langle f_1(x), f_2(x)\rangle_x+|f_2(x)|_x^2)^2\,d\mu$$

while $\|f\|_4^4$ is given by the same expression with the minus sign replaced by a plus sign. Thus $\|Jf\|_4^4=\|f\|_4^4$ is equivalent to

$$\int \operatorname{Re}\langle f_1(x), f_2(x)\rangle_x(|f_1(x)|_x^2+|f_2(x)|_x^2)\,d\mu=0.$$

But (2.1) for $p=4$ yields

$$\int \operatorname{Re}\langle f_1(x), f_2(x)\rangle_x\,|f_1(x)|_x^2\,d\mu=0$$

the same condition for $I-T$ yields

$$\int \operatorname{Re}\langle f_1(x), f_2(x)\rangle_x\,|f_2(x)|_x^2\,d\mu=0.$$

Thus J is an isometry on L^4.

Now a theorem of Banach and Lamperti [6] gives a description of the isometries on L^p for $p\neq 2$ for functions, and it easily extends to sections of vector bundles as well: we must have $Jf(x)=\lambda(x)\,h(x)(f\circ\psi)(x)$, where $\psi\colon\mathscr{F}\to\mathscr{F}$ is a σ-field homomorphism, h is a measurable function from X to the unitary operators on the fibre, and $\lambda(x)$ is a nonnegative measurable function on X satisfying

$$\int_{\psi^{-1}(A)}\lambda(x)^p\,d\mu=\mu(A) \tag{2.4}$$

for each $A\in\mathscr{F}$. But if (2.4) is to hold for $p=4$ and $p=2$ (given the form of J this is necessary for J to be an L^2 isometry) we must have $\lambda(x)\equiv 1$ a.e. and ψ must be measure preserving. From $J^2=I$ we deduce that ψ is an involution and $h\circ\psi(x)\,h(x)=I$ a.e.

Conversely, it is straightforward to verify that if J has the given form it is an involutive isometry on all L^p and that $T=\frac{1}{2}(I+J)$ and $T=\frac{1}{2}(I-J)$ are contractive projections on all L^p. Q.E.D.

Example. Let μ be a probability measure, and for scalar-valued functions consider the orthogonal projection onto the constants, $Tf(x)=\int f\,d\mu$. Then T is a contractive projection on all L^p. However, $I-T$ is easily seen not to be a contraction on L^1 (unless the measure consists of one or two atoms) hence it is not contractive on any L^p for $p\neq 2$.

3. The Heat Semigroup on Differential Forms

Let M be a complete Riemannian manifold of dimension n, and for $0 \leqslant k \leqslant n$ let $\Lambda^k M$ denote the bundle of differential k-forms, $\Delta = -d\delta - \delta d$ the Hodge–de Rham Laplacian (we choose the sign convention that makes Δ a negative operator), and $e^{t\Delta}$ the heat semigroup. At least on L^2 sections of $\Lambda^k M$, the heat semigroup is unique and contractive (see [1] or [11]). In analogy with the case of functions ($k=0$), we ask if the heat semigroup is L^p contractive for other values of p. As indicated in [11], we cannot expect the answer to be always "yes." In fact we will see that the answer is often "no," especially when the de Rham cohomology in dimension k is nontrivial.

The connection with cohomology is very simple: if $e^{t\Delta}$ were L^p contractive then by taking the limit as $t \to \infty$ we would obtain that the Kodaira projection operator T onto the harmonic k-forms is L^p contractive.

THEOREM 3.1. *Suppose M is compact, has nontrivial cohomology in dimension k, and $e^{t\Delta}$ on k-forms is L^p contractive for some $p \neq 2$. Then the space of harmonic k-forms is closed under $f(x) \to f(x)/|f(x)|$.*

Proof. This is an immediate consequence of Corollary 2.3 applied to the Kodaira projection. Since the manifold is compact the existence of the common dense domain is obvious. Q.E.D.

Thus, if we can show that there do not exist harmonic k-forms of constant absolute value, we can conclude immediately that the heat semigroup on k-forms is not contractive on any L^p, $p \neq 2$. In a number of cases it is possible to do this on purely local grounds. In dimension 2 the results are most decisive.

LEMMA 3.2. *On a nonflat 2-manifold there do not exist local nonzero harmonic 1-forms of constant absolute value.*

Proof. By introducing isothermal coordinates we may make the metric locally conformally flat, $g_{jk}(x) = h^2(x)\,\delta_{jk}$. Now if ω is a 1-form satisfying $d\omega = 0$ and $\delta\omega = 0$, then locally $\omega = df$, where f is an ordinary harmonic function. The condition $|\omega| = c$ just says $h^{-2}\,|\nabla f|^2 = c$, where $|\nabla f|^2 = (\partial f/\partial x^1)^2 + (\partial f/\partial x^2)^2$. But $(\partial f/\partial x^1) - i(\partial f/\partial x^2) = F$ is holomorphic, and from $h^2 = c^{-1}\,|F|$ it is straightforward to show that the metric is flat. Q.E.D.

For our next result we need to recall the definition of the *curvature operator* ρ as in Gallot and Meyer [3]. At each point on the manifold, the curvature operator is an operator from 2-forms to 2-forms (at the point) $\rho(\omega)_{kl} = R^{ij}{}_{lk}\,\omega_{ij}$. By the symmetry properties of the curvature tensor this is a symmetric operator, so it makes sense to say that ρ is positive definite, i.e.,

$$R^{ijlk}\omega_{ij}\omega_{kl} > 0 \tag{3.1}$$

for all $\omega \neq 0$. Note that this is a stronger statement than sectional curvature being strictly positive, which is just (3.1) for all ω of the form $\omega_{ij} = a_i b_j - a_j b_i$ (by the structure theory of the Lie algebra $so(n)$ the general skew-symmetric matrix is a sum of $[n/2]$ such special matrices).

LEMMA 3.3. *There are no nonzero harmonic k-forms of constant absolute value if either* (a), $k=1$ *and the Ricci curvature is strictly positive at one point or* (b), $k>1$ *and the curvature operator is positive definite at one point.*

Proof. The proof is based on the Bochner–Lichnerowicz formula [7, p. 3],

$$-\langle \Delta\omega, \omega\rangle = -\tfrac{1}{2}\Delta(|\omega|^2) + |\nabla\omega|^2 + F(\omega) \tag{3.2}$$

pointwise for k-forms ω. Of course if $\Delta\omega = 0$ and $|\omega| = c$ the first two terms vanish and so we obtain $F(\omega) \leqslant 0$. Now when $k=1$, $F(\omega)$ is just the Ricci curvature in the ω direction so we obtain (a). For $k>1$, Gallot and Meyer [3] show that $F(\omega)$ is expressible in terms of the curvature operator in such a way that $F(\omega) \leqslant 0$ contradicts the positive-definiteness. Q.E.D.

COROLLARY 3.4. *Let M be a compact Riemannian manifold with nontrivial real cohomology in dimension k. Then the heat semigroup for k-forms is not contractive on* L^p *for any* $p \neq 2$ *in the following cases*:

(i) $n=2$ *and the manifold is not flat,*

(ii) $k=1$ *and the Ricci curvature is strictly positive at one point,*

(iii) *the curvature operator is positive definite at one point.*

Next we consider some results in the other direction.

THEOREM 3.5. (a) *If the Ricci curvature is everywhere nonnegative, then the heat semigroup for* 1*-forms is* L^p *contractive for all p,* $1 \leqslant p \leqslant \infty$.

(b) *If the curvature operator is everywhere nonnegative, then the heat semigroup for k-forms is* L^p *contractive for all p,* $1 \leqslant p \leqslant \infty$.

Remark. There is no contradiction between this and the previous result, because if the Ricci curvature is everywhere nonnegative and strictly positive at one point, then the cohomology in dimension 1 must be trivial, and similarly if the curvature operator is everywhere nonnegative and positive at one point, there is no nontrivial cohomology [3]. On the other hand, it is not hard to construct manifolds with nonnegative curvature operator and nontrivial cohomology (in fact, symmetric spaces). Samelson [10] and Helgason [4] show that every homogeneous space of a compact Lie group has nonnegative sectional curvature, and we conjecture that even the curvature operator must be nonnegative (Gallot and Meyer [3] prove this for compact symmetric spaces).

Proof. (a) The Weitzenböck formula for 1-forms is simply

$$\Delta\omega = \Delta_0\omega - R^i{}_j\omega_i,$$

where Δ_0 is the Bochner Laplacian on tensors (trace of the second covariant derivative). Now the Bochner Laplacian Δ_0 generates a contraction semigroup on all L^p, $1 \leqslant p \leqslant \infty$. This is proved in [11] (the restriction $\frac{3}{2} \leqslant p \leqslant 3$ there is easily removed), and it is pointed out in [5] and [8] that it is an immediate consequence of the L^p contractivity of the heat semigroup for functions. The nonnegativity of the Ricci curvature means $-R^{ij}\omega_i\omega_j$ generates a contractive semigroup on all L^p, and the result follows by the Trotter product formula.

(b) To prove that Δ generates a contractive semigroup on L^p we have to establish the dissipative estimate

$$\langle |u|^{p-2}\, u, \Delta u\rangle \leqslant 0 \tag{3.3}$$

for k-forms $u \in \mathscr{D}$ and show that there are no positive eigenvalues for Δ with $L^{p'}$ eigenforms [9, pp. 240, 330; or 11, Lemma 3.3]. The key idea of the proof is the integrated Weitzenböck formula of Gallot and Meyer [2] (formula (0.8) polarized):

$$-\langle v, \Delta u\rangle = \langle \nabla v, \nabla u\rangle + Q(v, u), \tag{3.4}$$

where Q is a quadratic integral that is nonnegative if the curvature operator is everywhere nonnegative.

Now applying (3.4) with $v = |u|^{p-2}\, u$ (after a simple limiting argument since v may not be C^∞) we obtain

$$\begin{aligned} -\langle |u|^{p-2}\, u, \Delta u\rangle = {} & \langle \nabla(|u|^{p-2}\, u), \nabla u\rangle \\ & + Q(|u|^{p-2}\, u, u). \end{aligned} \tag{3.5}$$

But $-\langle |u|^{p-2}\, u, \Delta_0 u\rangle = \langle \nabla(|u|^{p-2}u), \nabla u\rangle \geqslant 0$ because the Bochner Laplacian is dissipative, and $Q(|u|^{p-2}\, u, u) \geqslant 0$ by the nonnegativity of the curvature operator. This establishes (3.3). Finally, if $\Delta u = \lambda u$ with $u \in L^{p'}$ and $\lambda > 0$ then (3.5) would yield an immediate contradiction. Since u does not have compact support we do not have (3.5) immediately, and a more delicate limiting argument is required, which is given in [11, Lemma 3.2]. This argument requires $p' < 3$, and thus establishes the L^p contractivity for $p > \frac{3}{2}$, and by duality we obtain the result for all L^p. Q.E.D.

Remark. In view of (3.5) it would seem likely that we could prove directly that if the curvature operator is sufficiently negative then the dissipative estimate is false, hence L^p contractivity fails, without involving cohomology. However, we have not succeeded in doing this.

Next we compute an explicit example of a manifold in which the heat semigroup is not even bounded on L^p, for any $t > 0$ or $p \neq 2$. Unfortunately the manifold is not connected. One might hope to get a connected example by joining up the components by long skinny tubes, but we have not been able to give an argument for this.

LEMMA 3.6. *Let M be the unit disc $r^2 = x^2 + y^2 < 1$ in the plane with metric $g_{jk}(x, y) = (1 - r^2)^{-\alpha} I$ with $\alpha \geqslant 2$. Then M is complete and contains no L^2 harmonic 1-forms that are in L^p if $1 \leqslant p \leqslant 2(1 - \alpha^{-1})$.*

Proof. Clearly M is complete if and only if the radial spokes have infinite length, and this amounts to $\int_0^1 (1 - r^2)^{-\alpha/2}\, dr = +\infty$ which is the condition $\alpha \geqslant 2$. Every harmonic 1-form can be written df, where $f = \sum_1^\infty a_n z^n + b_n \bar{z}^n$. Now

$$\|df\|_p^p = \int \left(g^{jk} \frac{\partial f}{\partial x^j} \frac{\partial f}{\partial x^k} \right)^{p/2} \sqrt{g}$$

so

$$\|df\|_2^2 = \iint |\nabla f(x, y)|^2\, dx\, dy$$

and $df \in L^2$ if and only if $\sum_1^\infty n^2(|a_n|^2 + |b_n|^2) < \infty$. However, if $df \in L^p$ this would imply $d(a_n z^n + b_n \bar{z}^n) \in L^p$ for each n, but the r-integration in $\|d(a_n z^n + b_n \bar{z}^n)\|_p^p$ will be of the order

$$\int_0^1 (1 - r^2)^{-\alpha(1 - p/2)}\, r^{(n-1)p} r\, dr$$

which diverges if $p \leqslant 2(1 - \alpha^{-1})$ unless $a_n = b_n = 0$. Q.E.D.

THEOREM 3.7. *There exists a (not necessarily connected) Riemannian 2-manifold M, each connected component of which is complete, such that the heat semigroup for 1-forms is not bounded on L^p for any $p \neq 2$ and any $t > 0$.*

Proof. Let $M_{\alpha,1}$ denote the manifolds in Lemma 3.6. Let F be a C^∞ 1-form of compact support. Then $\lim_{t\to\infty} e^{t\Delta}F$ will be an L^2 harmonic 1-form, and it is easy enough to guarantee that this limit is nonzero. Since by the lemma these 1-forms are not in L^p for $1 \leqslant p \leqslant 2(1-\alpha^{-1})$, we cannot have $\|e^{t\Delta}F\|_p \leqslant C\,\|F\|_p$ for all t for such p.

Next let $M_{\alpha,s}$ denote the dilation of $M_{\alpha,1}$ by a factor s. $M_{\alpha,s}$ is the unit disc in the plane with metric $g_{jk}(x, y) = s^{-1}(1-r^2)^{-\alpha} I$. It is easy to see that this change simply rescales the time parameter t in the heat semigroup. Thus for every p in $1 \leqslant p < 2$, every $t > 0$ and every $c < \infty$ there exists α and s such that $\|e^{t\Delta}F\|_p \leqslant c\,\|F\|_p$ is false for $M_{\alpha,s}$. Finally by taking a countable disjoint union of such manifolds we can contradict $\|e^{t\Delta}F\|_p \leqslant c\,\|F\|_p$ for all $1 \leqslant p < 2$ all $c > 0$ and all rational t (hence all real $t > 0$). By duality we cannot have boundedness for $2 < p \leqslant \infty$ either. Q.E.D.

Remark. If the curvature and its first two derivatives are bounded, Lohoué [8] has shown that the heat semigroup on differential forms is bounded on L^p for $1 < p < \infty$ with a bound exponential in t. In our example the curvature is clearly unbounded.

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